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BIOGRAPHY.

PAFNUTIJ LVOVITSCH TCHEBYCHEV.

BY DR. GEORGE BRUCE HALSTED.

OF Russian mathematicians, second only to Lobachevsky should be ranked Pafnutij Lvovitsch Tchebychev. Born in Russia in 1821 and formerly professor at the University of St. Petersburg, he reached deservedly the very highest scientific honors, being privy councillor, the representative of applied mathematics in the Imperial Academy of St. Petersburg, in 1860 made member of the famous Section I.—Geometrie, of the French *Academie des Sciences*, and afterward *Associe Etranger*, the highest honor attainable by a foreigner.

His best known work is the justly celebrated *Memoire Sur les Nombres Premiers*, (Academie imperiale de Saint-Petersbourg. 1850), where he established the existance of limits within which the sum of the logarithms of the primes inferior to a given number must be comprised. This memoir is given in *Liouville's Journal*, 1852, pp. 366-390.

Sylvester afterward contracted Tchebychev's limits; but the original paper remains highly remarkable, especially as it depends on very elementary considerations. In this respect it is in striking contrast to the equally marvelous paper of the lamented Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Groesse* [einer Untersuchung ueber die Haeufigkeit der Primzahlen], presented to the Berlin *Academie* in 1859.

Tchebychev had in 1848 presented a paper with this very title to the St. Petersburg Academie: Sur la totalite des nombres premiers inferieurs a une limite donnee. (Given in Liouville's Journal, 1852, pp. 341-365.) Riemann speaks of the interest long bestowed on this subject by Gauss and Dirichlet,

but makes no mention of Tchebychev. However Sylvester speaks of "his usual success in overcoming difficulties insuperable to the rest of the world."

But though best known for his work in the most abstract part of mathematics, in reality Tchebychev was of an eminently practical turn of mind. Thus it was his work "*Theorie des Mecanismes connus sous le nom de parallelogrammes* (Memoires des savants etrangers, Tom. VII.) which led him to the elaborate dissertation *Sur les questions de Minima qui se rattachent a la representation approximative des fonctions*, 91 quarto pages in Memoires de l'Academie Imperiale des sciences de Saint Petersbourg, 1858. While the variable x remains in the vicinity of one same value we can represent with the greatest possible approximation any function $f(x)$, of given form, by the principles of the differential calculus. But this is not the case if the variable x is only required to remain within limits more or less extended. The essentially different methods demanded by this case, which is just the one met in practice, are developed in this memoir. The same line of thought led to his connection with a subject which has since found a place even in elementary text-books, namely rectilineal motion by linkage. He invented a three-bar linkage, which is called Tchebychev's parallel motion, and gives an extraordinarily close approximation to exact rectilineal motion; so much so that in a piece of apparatus exhibited by him in the London Loan Collection of Scientific Apparatus, a plane supported on a combination of two of his parallel motion linkages seemed to have a strictly horizontal movement, though its variation was double that of the tracer in the simple parallel motion.

Tchebychev long occupied himself with attempting to solve the problem of producing exact rectilineal motion by linkage, until he became convinced that it was impossible and even strove long to find a proof of that impossibility. What must have been his astonishment then, when a freshman student of his own class, named Lipkin, showed him the long-sought conversion of circular into straight motion.

Tchebychev brought Lipkin's name before the Russian government, and secured for him a substantial reward for his supposed original discovery.

And perhaps it was independent, but it had been found several years previously by a French lieutenant of engineers, Peaucellier, and first published by him in the form of a question in the '*Annales de Mathematique* in 1864. When Tchebychev was on a visit to London, Sylvester inquired after the progress of his proof of the impossibility of exact parallel motion, when the Russian announced its double discovery and made a drawing of the cell and mounting. This Sylvester happened to show to Manuel Garcia, inventor of the laryngoscope, and the next day received from him a model constructed of pieces of wood fastened with nails as pivots, which, rough as it was, worked perfectly. Sylvester exhibited this to the Philosophical Club of the Royal Society and in the Athenaeum Club, where it delighted Sir Wm. Thomson, now Lord Kelvin, and led to the extraordinary lecture *On recent Discoveries in Mechanical Conversion of Motion*, delivered by Sylvester before the Royal Institution on January 23, 1874. This in turn led to Kempe's remarkable developement of

the subject, and to Hart's discovery of a five-bar linkage which does the same work as Peaucellier's of seven. Henceforth Peaucellier's Cell and Hart's Contraparallelogram will take their place in our text-books of geometry, and straight lines can be drawn without begging the question by assuming first a straight edge or ruler as does Euclid. Thus Kempe's charming book, *How to draw a straight line*, is a direct outcome of Tchebychev's sketch for Sylvester, to whom, in parting, he used the characteristic words: "Take to kinematics, it will repay you; it is more fecund than geometry; it adds a fourth dimension to space." As might perhaps have been expected, the immortal Lobachevsky found in his compatriot a devoted admirer. Not only was Tchebychev an active member of the Committee of the Lobachevsky-fund, but he took the deepest interest in all connected with the spread of the profound ideas typified in the Non-Euclidean geometry.

Knowing this, Vasiliev in his last letter asked that a copy of my translation of his Address on Lobachevsky be forwarded to the great man. His active participation in scientific assemblies is also worthy of note; for example at the 'Congres de l'Association francaise pour l' avancement des sciences, Lyon' he read two interesting papers, *Sur les valeurs limites des integrales*, and *Sur les quadratures*, both afterwards published in *Liouville's Journal*.

LAGRANGE'S GENERALIZED EQUATIONS OF MOTION.

By F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Lagrange's celebrated equations of motion, as given in his *Mecanique Analytique*, consist of *a transformation* from Cartesian to generalized co-ordinates, *of the indeterminate equation of motion*. In order to avoid the unnecessary complication incident to the introduction of such indeterminate variations, as δx, δy, δz, etc., in the derivation of these equations of motion, we must have recourse to the Cartesian equations of the unconstrained motion of particles.

Assume $(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n)$ as the rectilineal-rectangular co-ordinates of the material particles $m_1, \ldots, m_n$; also, assume $(X_1, Y_1, Z_1), \ldots, (X_n, Y_n, Z_n)$ as the force-components of the particles. Make $(\psi_1, \phi_1, \theta_1), \ldots, (\psi_n, \phi_n, \theta_n)$ the generalized co-ordinates of the *respective* particles at any instant of time; that is, let ψ_1, ϕ_1, θ_1, be regarded as determinate functions of x_1, y_1, z_1, respectively,—or vice versa. This reciprocal determinateness is to be a characteristic of all the material particles. The velocity-components in

Cartesian co-ordinates are $(dx_1 / dt, dy_1 / dt, dz_1 / dt)$, etc.; while the corresponding generalized velocity-components are $(d\psi_1 / dt, d\phi_1 / dt, d\theta_1 / dt)$, etc. Similarly the acceleration-components in Cartesian co-ordinates are $(d^2x_1 / dt^2, d^2y_1 / dt^2, d^2z_1 / dt^2)$, etc.: while the corresponding generalized acceleration-components are $(d^2\psi_1 / dt^2, d^2\phi_1 / dt^2, d^2\theta_1 / dt^2)$, etc.

From well-known principles of the differential calculus,

$$\frac{dx_1}{dt} = \frac{dx_1}{d\psi_1}\cdot\frac{d\psi_1}{dt} + \frac{dx_1}{d\phi_1}\cdot\frac{d\phi_1}{dt} + \frac{dx_1}{d\theta_1}\cdot\frac{d\theta_1}{dt}\ldots.(A_1),$$

$$\frac{dy_1}{dt} = \frac{dy_1}{d\psi_1}\cdot\frac{d\psi_1}{dt} + \frac{dy_1}{d\phi_1}\cdot\frac{d\phi_1}{dt} + \frac{dy_1}{d\theta_1}\cdot\frac{d\theta_1}{dt}\ldots.(A_2),$$

$$\frac{dz_1}{dt} = \frac{dz_1}{d\psi_1}\cdot\frac{d\psi_1}{dt} + \frac{dz_1}{d\phi_1}\cdot\frac{d\phi_1}{dt} + \frac{dz_1}{d\theta_1}\cdot\frac{d\theta_1}{dt}\ldots.(A_3).$$

Since the kinetic energy of the particle m_1 represented by the Cartesian co-ordinates (x_1, y_1, z_1) becomes

$$K_1 = \tfrac{1}{2}m_1[(dx_1 / dt)^2 + (dy_1 / dt)^2 + (dz_1 / dt)^2]\ldots.(a),$$

it is obvious [substituting the square of (A_1), of (A_2), and of (A_3), in (a)] that the corresponding kinetic energy expressed in terms of generalized co-ordinates may be represented (according to the usual system of functional notation) by

$$T_1 = \tfrac{1}{2}\left[(\psi_1,\psi_1)\left(\frac{d\psi_1}{dt}\right)^2 + (\phi_1,\phi_1)\left(\frac{d\phi_1}{dt}\right)^2 + (\theta_1,\theta_1)\left(\frac{d\theta_1}{dt}\right)^2\right.$$

$$\left. + 2(\psi_1,\phi_1)\left(\frac{d\psi_1}{dt}\cdot\frac{d\phi_1}{dt}\right) + 2(\phi_1,\theta_1)\left(\frac{d\phi_1}{dt}\cdot\frac{d\theta_1}{dt}\right) + 2(\theta_1,\psi_1)\left(\frac{d\theta_1}{dt}\cdot\frac{d\psi_1}{dt}\right)\right]\ldots(a').$$

The *coefficients* in (a'), *inclosed by the smaller parentheses*, are homogeneous functions of the co-odinates. These coefficients are determinable from system-conditions. With respect to these coefficients, the necessary condition is that they must give a finite and positive value of T_1, for whatever value may be assigned to the variables.

If $(\delta x_1, \delta y_1, \delta z_1), \ldots, (\delta x_n, \delta y_n, \delta z_n)$ represent the components of any infinitely small motions possible without breaking the conditions of the material system of which $m_1, \ldots, m_n$ are the component particles: then *the work done by the forces* whose components have already been specified, *upon the particle* m_1, becomes $W_1 = X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 \ldots.(b)$.

In order to deduce the expression for W_1 in terms of generalized co-ordinates, we have from the differential calculus:

$$\delta x_1 = \frac{dx_1}{d\psi_1}\delta\psi_1 + \frac{dx_1}{d\phi_1}\delta\phi_1 + \frac{dx_1}{d\theta_1}\delta\theta_1\ldots.(B_1),$$

$$\delta y_1 = \frac{dy_1}{d\psi_1}\delta\psi_1 + \frac{dy_1}{d\phi_1}\delta\phi_1 + \frac{dy_1}{d\theta_1}\delta\theta_1 \ldots . (B_2),$$

$$\delta z_1 = \frac{dz_1}{d\psi_1} + \frac{dz_1}{d\phi_1}\delta\phi_1 + \frac{dz_1}{d\theta_1}\delta\theta_1 \ldots . (B_3).$$

Multiplying (B_1) by X_1, (B_2) by Y_1, (B_3) by Z_1, and adding products, we have

$$X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 = \left(X_1\frac{dx_1}{d\psi_1} + Y_1\frac{dy_1}{d\psi_1} + Z_1\frac{dz_1}{d\psi_1}\right)\delta\psi_1$$

$$\left(X_1\frac{dx_1}{d\phi_1} + Y_1\frac{dy_1}{d\phi_1} + Z_1\frac{dz_1}{d\phi_1}\right)\delta\phi_1 + \left(X_1\frac{dx_1}{d\theta_1} + Y_1\frac{dy_1}{d\theta_1} + Z_1\frac{dz_1}{d\theta_1}\right)\delta\theta_1,$$

$= \Psi_1\delta\psi_1 + \Phi_1\delta\phi_1 + \Theta_1\delta\theta_1 \ldots (d_1)$, in which the coefficients of the indeterminate variations are the generalized force-components with respect to the particle m_1. According to D'Alembert's principle, the Cartesian *Equations of motion* in the order of the particles specified, become

$$X_1 = m_1\frac{d^2x_1}{dt^2},\ Y_1 = m_1\frac{d^2y_1}{dt^2},\ Z_1 = m_1\frac{d^2z_1}{dt^2}, \ldots .,$$

$$X_n = m_n\frac{d^2x_n}{dt^2},\ Y_n = m_n\frac{d^2y_n}{dt^2},\ Z_n = m_n\frac{d^2z_n}{dt^2}.$$

Multiplying these equations, respectively, by

$\frac{dx_1}{d\psi_1}, \frac{dy_1}{d\psi_1}, \frac{dz_1}{d\psi_1}, \ldots ., \frac{dx_n}{d\psi_n}, \frac{dy_n}{d\psi_n}, \frac{dz_n}{d\psi_n}$; and then adding the products, we have the equations,

$$\Psi = m_1\left(\frac{d^2x_1}{dt^2} \cdot \frac{dx_1}{d\psi_1} + \frac{d^2y_1}{dt^2} \cdot \frac{dy_1}{d\psi_1} + \frac{d^2z_1}{dt^2} \cdot \frac{dz_1}{d\psi_1}\right) + \ldots .$$

$$+ m_n\left(\frac{d^2x_n}{dt^2} \cdot \frac{dx_n}{d\psi_n} + \frac{d^2y_n}{dt^2} \cdot \frac{dy_n}{d\psi_n} + \frac{d^2z_n}{dt^2} \cdot \frac{dz_n}{d\psi_n}\right) \ldots . (c_1).$$

After performing similar operations, we have the equations:

$$\Phi = m_1\left(\frac{d^2x_1}{dt^2} \cdot \frac{dx_1}{d\phi_1} + \frac{d^2y_1}{dt^2} \cdot \frac{dy_1}{d\phi_1} + \frac{d^2z_1}{dt^2} \cdot \frac{dz_1}{d\phi_1}\right) + \ldots .$$

$$+ m_n\left(\frac{d^2x_n}{dt^2} \cdot \frac{dx_n}{d\phi_n} + \frac{d^2y_n}{dt^2} \cdot \frac{dy_n}{d\phi_n} + \frac{d^2z_n}{dt^2} \cdot \frac{dz_n}{d\phi_n}\right) \ldots . (c_2),$$

and $\Theta = m_1\left(\frac{d^2x_1}{dt^2} \cdot \frac{dx_1}{d\theta_1} + \frac{d^2y_1}{dt^2} \cdot \frac{dy_1}{d\theta_1} + \frac{d^2z_1}{dt^2} \cdot \frac{dz_1}{d\theta_1}\right) + \ldots .$

$$+ m_n\left(\frac{d^2x_n}{dt^2} \cdot \frac{dx_n}{d\theta_n} + \frac{d^2y_n}{dt^2} \cdot \frac{dy_n}{d\theta_n} + \frac{d^2z_n}{dt^2} \cdot \frac{dz_n}{d\theta_n}\right) \ldots . (c_3).$$

Taken in its utmost generality; that is, in case of *isolated* and *con-*

servative Material-systems, (a) becomes $K=\Sigma\frac{1}{2}m[(dx\,/\,dt)^2+(dy\,/\,dt)^2+(dz\,/\,dt)^2]\ldots(a_1)$; also, from ($a'$), in case of analogous systems,

$$T=\frac{1}{2}\Big[(\psi,\psi)\Big(\frac{d\psi}{dt}\Big)^2+(\phi,\phi)\Big(\frac{d\phi}{dt}\Big)^2+(\theta,\theta)\Big(\frac{d\theta}{dt}\Big)^2$$

$$+2(\psi,\phi)\Big(\frac{d\psi}{dt}\cdot\frac{d\phi}{dt}\Big)+2(\phi,\theta)\Big(\frac{d\phi}{dt}\cdot\frac{d\theta}{dt}\Big)+2(\theta,\psi)\Big(\frac{d\theta}{dt}\cdot\frac{d\psi}{dt}\Big)\Big]\ldots(a'_1).$$

Supposing $dx_1\,/\,dt$ to be a function of the Cartesian co-ordinates, and also a function of the generalized velocity-components, we have from the differential calculus:

$$\frac{d^2x_1}{dt^2}\cdot\frac{dx_1}{d\psi_1}=\frac{d}{dt}\Big(\frac{dx_1}{dt}\cdot\frac{dx_1}{d\psi_1}\Big)-\frac{dx_1}{dt}\cdot\frac{d}{dt}\Big(\frac{dx_1}{d\psi_1}\Big),$$

$$=\frac{d}{dt}\Big[\frac{dx_1}{dt}\Big(\frac{dx_1}{dt}\Big/\frac{d\psi_1}{dt}\Big)\Big]-\frac{dx_1}{dt}\Big(\frac{dx_1}{dt}\Big/d\psi_1\Big),$$

$$=\frac{d}{dt}\Big[\frac{1}{2}\text{ of }\frac{d[(dx_1\,/\,dt)^2]}{d(d\psi_1\,/\,dt)}\Big]-\frac{1}{2}\text{ of }\frac{d[(dx_1\,/\,dt)^2]}{d\psi_1}\ldots(c).$$

For *every* term of (c_1), (c_2), and (c_3) *the proper expression* can be written by analogy, from the right-hand member of (c).

Transforming (c_1), (c_2), and (c_3), by means of (c) *generally applied* and then using T for the kinetic energy of the system, we have

$$\frac{d}{dt}\Big(dT\Big/\frac{d\psi}{dt}\Big)-\frac{dT}{d\psi}=\Psi,\ \frac{d}{dt}\Big(dT\Big/\frac{d\phi}{dt}\Big)-\frac{dT}{d\phi}=\Phi,$$

and $\frac{d}{dt}\Big(dT\Big/\frac{d\theta}{dt}\Big)-\frac{dT}{d\theta}=\Theta,$

which are Lagrange's equations of motion in terms of generalized co-ordinates.

With respect to any isolated and conservative material-system, the generalization of (b_1) gives $\Sigma(X\delta x+Y\delta y+Z\delta z)=\Psi\delta\psi+\Phi\delta\phi+\Theta\delta\theta\ldots(d)$.

The potential energy of a conservative system is a function of the co-ordinates by which the different positions of the various parts of such a system are specified. With reference to the configuration which an isolated and conservative material-system has at any instant, the potential energy represents the amount of work required to bring the system to that configuration against its mutual forces during the passage of the system from any one chosen configuration to the configuration referred to at the time. Hence if an aggregation of moving particles constitutes an isolated and conservative material-system whose potential energy in the configuration specified by the Cartesian co-ordinates x, y, z) is represented by V, we must have $\delta V=-\Sigma(X\delta x+Y\delta y+Z\delta z)\ldots(e)$.

From (d) and (e), $\Psi\delta\psi + \Phi\delta\phi + \Theta\delta\theta = -\delta V \ldots (f)$.

$$\therefore \quad \Psi = -\frac{dV}{d\psi}, \quad \Phi = -\frac{dV}{d\phi}, \text{ and } \Theta = -\frac{dV}{d\theta}.$$

Therefore the Lagrangian equations of motion may be written:

$$\frac{d}{dt}\left(dT \Big/ \frac{d\psi}{dt}\right) = \frac{dT}{d\psi} - \frac{dV}{d\psi}, \quad \frac{d}{dt}\left(dT \Big/ \frac{d\phi}{dt}\right) = \frac{dT}{d\phi} - \frac{dV}{d\phi},$$

$$\text{and } \frac{d}{dt}\left(dT \Big/ \frac{d\theta}{dt}\right) = \frac{dT}{d\theta} - \frac{dV}{d\theta}.$$

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

BY GEORGE BRUCE HALSTED, A. M., (Princeton) Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the February Number].

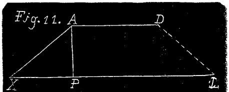
PROPOSITION XIV. *The hypothesis of obtuse angle is inconsistent with Euclid's assumption: Two straight lines cannot enclose a space.*

Proof. From the hypthesis of obtuse angle, assumed as true, [and the first 28 propositions of Euclid], we have now deduced the truth of Euclid's Postulatum; that two straights will meet each other in some point toward those parts, toward which a certain straight, cutting them, makes two internal angles, of whatever kind, less than two right angles.

But this Postulatum holding good, on which Euclid supports himself after the twenty-eighth proposition of his first book, it is manifest to all Geometers, that the hypothesis of right angle alone is true, nor any place left for the hypothesis of obtuse angle. Therefore the hypothesis of obtuse angle is inconsistent with Euclid's assumption. Quod erat demonstrandum.

Otherwise, and more immediately.

Since from the hypothesis of obtuse angle we have demonstrated (P. IX.) that two (fig. 11.) acute angles of the triangle APX, right-angled at P, are greater than one right angle; it follows that an acute angle PAD may be assumed such, that together with the aforesaid two acute angles it makes up two right angles. But then the straight AD must (by the preceding proposition,

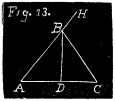


joined to the hypothesis of obtuse angle) at length meet with this PL, or XL, regard being had to the secant, or incident AP; which is manifestly absurd against Eu. I. 17, if we regard the secant or incident AX.

PROPOSITION XV. *By any triangle* ABC, *of which the three angles* (*fig. 13.*) *are together equal to, or greater, or less than two right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle.*

Proof. For anyhow two angles of this triangle, as suppose A, and C, will be acute, because of Eu. I. 17. Wherefore the perpendicular, let fall from the apex of the remaining angle B upon this AC, will cut this AC (Eu. I. 17.) in some intermediate point D.

Fig. 13.



If therefore three angles of this triangle ABC are supposed equal to two right angles, it follows that all the angles of the triangles ADB, CDB will be together equal to four right angles, because of the two additional right angles at the point D. This holding good, now of neither of the said triangles, as suppose ADB, will the three angles together be less, or greater than two right angles; for thus viceversa the three angles together of the other triangle would be greater, or less than two right angles. Wherefore (P. IX.) from one triangle indeed would be established the hypothesis of acute angle, and from the other the hypothesis of obtuse angle; which is contrary to P. VI. and P. VII.

Therefore the three angles together of either of the aforesaid triangles will be equal to two right angles; and therefore (P. IX.) is established the hypothesis of right angle Quod erat primo loco demonstrandum.

But if however the three angles of the proposed triangle ABC are taken greater than two right angles; now of the two triangles ADB, CDB all the angles together will be greater than four rignt angles, because of the two additional right angles at the point D.

This holding good; now of neither of the said triangles will the three angles together be precisely equal to, or less than two right angles; for thus viceversa the three angles of the other triangle would be together greater than two right angles. Wherefore (P.IX.) from one triangle indeed would be established the hypothesis either of right angle or of acute angle, and from the other the hypothesis of obtuse angle, which is contrary to P.V, P.VI, and P.VII.

Therefore the three angles together of either of the aforesaid triangles will be greater than two right angles; and therefore (P.IX.) is established the hypothesis of obtuse angle. Quod erat secundo loco demonstrandum.

But finally. If the three angles of the proposed triangle ABC are taken less than two right angles, now of the two triangles ADB, CDB, all the angles together will be less than four right angles, because of the two additional right angles at the point D.

This holding good; now of neither of the said triangles will the three angles together be equal to, or greater than two right angles; for thus viceversa of the other triangle the three angles together would be less than two right angles. Wherefore (P.IX.) from one triangle indeed would be establish-

ed the hypothesis either of right angle or obtuse angle, and from the other the hypothesis of acute angle; which is contrary to P.V, P.VI, and P.VII.

Therefore the three angles together of either of the aforesaid triangles will be less than two right angles; and therefore (P.IX.) is established the hypothesis of acute angle. Quod erat tertio loco demonstrandum.

Accordingly by any triangle ABC, of which the three angles are equal to, or greater, or less than two right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle. Quod erat propositum.

COROLLARY. Hence, any one side of any proposed triangle being produced, as suppose AB to H, the external angle HBC will be (Eu.I.13.) either equal to, or less, or greater than the remaining internal and opposite angles together at the points A and C, according as is true the hypothesis of right angle, or obtuse angle, or acute angle. And inversely.

[To be continued.]

THE "IRREDUCIBLE CASE."

By J. K. ELLWOOD, A. M., Principal Colfax School, Pittsburg, Pa.

PROBLEM.—*To extract the cube root of* $a \pm \sqrt{-b}$.

Put $\sqrt[3]{a+\sqrt{-b}}=m+n$, and $\sqrt[3]{a-\sqrt{-b}}=m-n$.

Then $a+\sqrt{-b}=m^3+3m^2n+3mn^2+n^3$, and $a-\sqrt{-b}=m^3-3m^2n+3mn^2-n^3$. Hence $a=m^3+3mn^2$, and $\sqrt{-b}=3m^2n+n^3$.

Example 1. Find the cube root of $9+25\sqrt{-2}$.

Here $a=m^3+3mn^2=9=3^3-18$. Hence $3mn^2=-18$, and $n=\sqrt{-2}$.

To verify these values of m and n substitute them in $\sqrt{-b}=3m^2n+n^3=25\sqrt{-2}$. Doing this we have $27\sqrt{-2}+(-2\sqrt{-2})=25\sqrt{-2}$.

$\therefore$ $3+\sqrt{-2}$ is the required root. When the substituted values of m and n do not give the second term they are not correct, and other values must be found by trial.

Example 2. Find the cube root of $2\sqrt{11}+30\sqrt{-3}$. Here $a=m^3+3mn^2=2\sqrt{11}=(\sqrt{11})^3-9\sqrt{11}$.

Hence $3mn^2=-9\sqrt{11}$, and $n=\sqrt{-3}$. Since these values of m and n substituted in $3m^2n+n^3$ give $\sqrt{-b}=30\sqrt{-3}$, the root is $\sqrt{11}+\sqrt{-3}$.

This method frequently enables us to simplify Cardan's formula for cubics in what is called the "irreducible case", said formula being

$$x=\sqrt[3]{q+\sqrt{(q^2-p^3)}}+\sqrt[3]{q-\sqrt{(q^2-p^3)}}.$$

1. * $x^3-22x-24=0$. Here $p=\frac{22}{3}$, $q=12$.

Then $x=\sqrt[3]{12+\frac{26}{3}\sqrt{(-\frac{10}{3})}}+\sqrt[3]{12-\frac{26}{3}\sqrt{(-\frac{10}{3})}}$.

Now, $a=m^3+3mn^2=12=-2^3+20$. $3mn^2=20$, $n=\sqrt{-\frac{10}{3}}$.

$-2\pm\sqrt{-\frac{10}{3}}$ are the roots. Then $x=(-2+\sqrt{-\frac{10}{3}})+(-2-\sqrt{-\frac{10}{3}})=-4$.

2. $x^3-8x^2+19x-12=0$. Put $x=y+\frac{8}{3}$.

Then $y^3-\frac{7}{3}y+\frac{20}{27}=0$, where $p=\frac{7}{9}$, $q=-\frac{10}{27}$.

Then $y=\sqrt[3]{-\frac{10}{27}+\sqrt{(-\frac{243}{729})}}+\sqrt[3]{-\frac{10}{27}-\sqrt{(-\frac{243}{729})}}$

$=\sqrt[3]{-\frac{10}{27}+\frac{1}{3}\sqrt{-3}}+\sqrt[3]{-\frac{10}{27}-\frac{1}{3}\sqrt{-3}}$

Here $a=m^3+3mn^2=-\frac{10}{27}=(\frac{2}{3})^3-\frac{18}{27}$. Hence $n=\sqrt{-\frac{1}{3}}=\frac{1}{3}\sqrt{-3}$.

$\therefore$ $y=(\frac{2}{3}+\frac{1}{3}\sqrt{-3})+(\frac{2}{3}-\frac{1}{3}\sqrt{-3})=\frac{4}{3}$. Then $x=\frac{4}{3}+\frac{8}{3}=4$.

3. $x^3-12x^2+41x-42=0$. Put $x=y+4$.

Then $y^3-7y=6$, where $p=\frac{7}{3}$, $q=3$. Then $y=\sqrt[3]{3+\frac{10}{9}\sqrt{-3}}+\sqrt[3]{3-\frac{10}{9}\sqrt{-3}}$.

Here $a=3=(\frac{3}{2})^3-\frac{3}{8}$. $\therefore$ $n=\frac{1}{6}\sqrt{-3}$.

$\therefore$ $y=(\frac{3}{2}+\frac{1}{6}\sqrt{-3})+(\frac{3}{2}-\frac{1}{6}\sqrt{-3})=3$, and $x=3+4=7$.

* Maynard in his Key to Bonnycastle's Introduction, page 78, says this "can only be resolved by a table of sines, or by infinite series."

THEOREM 16 OF LOBATSCHEWSKY'S THEORY OF PARALLELS.

By JOHN N. LYLE, Ph. D., Professor of Natural Sciences, Westminster College, Fulton, Missouri

Says Lobatschewsky in his Theorem 16—"All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes—into *cutting* and *not-cutting*. The *boundary lines* of the one and the other class of those lines will be called *parallel to the given line*.

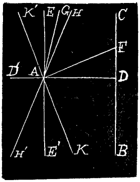


Fig. 1.

From the point A (Fig. 1) let fall upon the line BC the perpendicular AD, to which again draw the perpendicular AE. In the right angle EAD either will all all straight lines which go out from the point A meet the line DC, as for example AF, or some of them, like the perpendicular AE, will not meet the line DC. In the uncertainty whether the perpendicular AE is the only line which does not meet DC, we will assume it may be possible that there are still other lines, for

example AG, which do not cut DC, how far soever they may be prolonged. In passing over from the cutting lines, as AF, to the not-cutting lines, as AG, we must come upon a line AH, parallel to DC, a boundary line, upon one side of which all lines AG are such as do not meet the line DC, while upon the other side every straight line AF cuts the line DC.

The angle HAD between the parallel HA and the perpendicular AD is called the parallel angle (angle of parallelism), which we will here designate by $\Pi(p)$ for $AD=p$."

Does Lobatschewsky class his boundary line AH among the *cutting* or the *not-cutting* lines? Evidently among the cutting lines, for under Theorem 33, referring to his equation $S'=se^{-x}$, he says -"We may here remark, that $S'=0$ for $x=\infty$, hence not only does the distance between two parallels decrease (Theorem 24), but with the pralongation of the parallels towards the side of the parallelism this at last wholly vanishes."

Agreeably to this assumption of Lobatschewsky let y be the point in space at which the decreasing distance between his parallel lines AH and DC wholly vanishes. According to Euclid's postulate 2 the terminated line Dy may be extended beyond the point y. If this is not permitted, Euclid's postulate 2 would be discredited in Lobatschewsky's geometry. Assume that Euclid's postulates hold everywhere in space. On the basis of that assumption we have the authority to locate any point as z beyond y on Dy extended. Then the point z is within the angle yAE. From z draw a straight line to the point A. This must be permitted, otherwise postulate 1 would be discredited in Lobatschewsky's geometry. Since z is within the angle yAE the straight line Az must fall between Ay and AE.

But since by Lobatschewsky's hypothesis no straight line between AHy and AE can meet DC produced, the line Az must fall between Ay and AD. That is, Az must lie on both sides of Ay at the same time. Says W. K. Clifford, an enthusiastic admirer of Lobatschewsky's Imaginary Geometry—"but the way things come out of one onother is quite lovely."

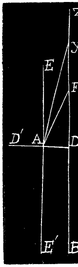


Fig. 2.

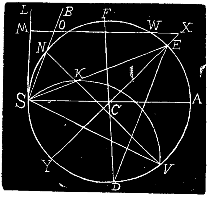
SOME FALLACIES OF AN ANGLE TRISECTOR.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, The University of Chicago.

Since 1860, Mr. L. S. Benson of New York City has labored to throw

discredit upon vital parts of mathematics. He easily trisects any angle by simple geometry and because the results following from his construction are not in harmony with results derived from Trigonometry, he quietly casts overboard as false all Trigonometry and with it the applied mathematics!

His trisection, however, really offers an interesting fallacy especially from a graphical standpoint. The *gist* of the construction given in his circulars "Mathematics out of joint," etc., is as follows:



To trisect any angle BSA between 60° and 90°, construct the right angle LSA and a circle with any point C on SA as center. Make arc $AW=60$ and draw $WM \parallel SA$. Take $WX=MO$ and draw diameter $XECY$. Draw chord $ED \parallel SB$ and diameter DF. Then arc NA is trisected by F and E. Proof: arcs NE, SD, FA are equal, also $FE=YD$, also $NF=SY=EA$. Draw chord $DV \parallel SE$. Describe circle SKV with radius $=SC$ through points S and V. Then arc $SN=$ arc SK, each being double the measure of $\angle NVS$; similarly, $\angle ESV$ makes arc $KV=$ arc EV; $\angle NVE$ makes arc $KV=$ arc NE. Since arcs $SKV=SDV$ and $KV-EV=SD$, then $SN=SK=DV$. $\therefore$ arc $SNE=$ arc EVD and finally arc $SY=$ arc YD. $\therefore$ arcs $NF=FE=EA$.

Since 3 arc $AW=180°=3$ (arc $AE+$ arc EW) $=$ arc $AN+3$ arc EW, we find arc $SN=3$ arc EW. Thus any angle $< 90°$ is trisected.

The step used by Mr. Benson which lacks proof is the fact that NV and SE intersect on the arc drawn through S and V with radius $=SC$. It can be shown by Trigonometry that this fact is true only for one particular angle $67\frac{1}{2}$°. But the error is so small that it is scarcely apparent in an ordinary-sized figure, even if drawn accurately. But with a larger figure and the favorable case of an angle near 60° or 90°, the error is distinct. The deception is increased by the fact that to a certain difference in the lengths of the arcs NF and FE corresponds a much smaller distance between K_1 (the intersection of NV and arc SV) and K_2 (the intersection of SE and arc SV). In "another proof," Mr. Benson attempts to prove that K_1 and K_2 coincide (in the point K of the figure) but lands immediately in a rediculous *argumento in circulo*.

ARITHMETIC.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

35. **Proposed by B. F. FINKEL, Professor of Mathematics in Kidder Institute, Kidder, Missouri.**

Between Sing-Sing and Tarry-Town, I met my worthy friend, John Brown,
And seven daughters, riding nags, and every one had seven bags:
In every bag were thirty cats, and every cat had forty rats,
Besides a brood of fifty kittens. All *but* the nags were wearing mittens!
Mittens, kittens—cats, rats—bags, nags—Browns,
How many were met between the towns?

[From *Mattoon's Common Arithmetic*].

II. **Solution by T. W. PALMER, Professor of Mathematics, University, Alabama.**

1. 8 = No. of Browns met.
2. $8=8\times1$ = No. of nags.
3. $112=16\times7$ = No. of bags, (each bag and each Brown had 7).
4. $3360=112\times30$ = No. of cats.
5. $134400=3360\times40$ = No. of rats.
6. $168000=3360\times50$ = No. of kittens.
7. $16=8\times2$ = No. of mittens worn by Browns.
8. $13440=3360\times4$ = No. of mittens worn by cats.
9. $537600=134400\times4$ = No. of mittens worn by rats.
10. $672000=168000\times4$ = No. of mittens worn by kittens.

1528944 = Browns + nags + bags + cats + rats + kittens + mittens.

NOTE—Mr. Horn in January Number has probably given correct solution, but the language of the example will admit of the above interpretation. *T. W. P.*

Remark on Solution of Number 35 by COOPER D. SCHMITT, Knoxville, Tennessee.

Mr. Horn's addition is not correct to obtain line numbered 7. I would suggest that as cats, rats and kittens have *four* legs each that *four* mittens be assigned to each, this will make the answer to the problem, 764488. I can not see Mr. Mattoon's interpretation of the problem.

38. **Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.**

What must be the thickness of a 36-inch shell, in order that it may weigh 1 ton: supposing a 13-inch shell to weigh 200 pounds, when two inches thick?

III. **Solution by P. S. BERG, Apple Creek, Ohio.**

$$\frac{13^3\pi}{6}-\frac{9^3\pi}{6}=\frac{1468\pi}{6}, \text{ solid contents of 13-inch shell.}$$

$$\frac{1468\pi}{6}\times10=\frac{14680\pi}{6}, \text{ solid contents of 36-inch shell}$$

$$\frac{36^3\pi}{6}-\frac{14680\pi}{6}=\frac{31976\pi}{6}, \text{ volume of hollow within 36-inch shell}$$

$$\sqrt[3]{\frac{31976\pi}{6}\div\frac{\pi}{6}}=31.736, \text{ diameter of hollow within 36-inch shell.}$$

$(36-31.736)\div 2=2.132$ in. thickness of shell.

This problem was solved with same result, by *Hon. Josiah H. Drummond, J. F. W. Scheffer, Frank Horn J. K. Ellwood, and Cooper D. Schmitt.*

39. Proposed by P. C. CULLEN, Superintendent of Schools, Brady, Nebraska.

A, B, and C start from same point at same time. A north at rate of three miles per hour, B east at rate of four miles and C west at rate of five miles per hour. B at end of two hours starts at such an angle as to intersect A. How long after starting must C start north-west in order to meet A and B at common point ?

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Maine, and J. W. WATSON, Middle Creek, Ohio.

Let x be the time after B turns till he meets A. The route of both is a right angle triangle with base 8; perpendicular $3x+6$, and hypotenuse $4x$. Hence, $16x^2=(3x+6)^2+64$, whence $x=7\frac{1}{7}$ or -2. But the -2 value makes them turn back and meet at point of starting. Let y=time before C turns. Then $7\frac{1}{7}+2-y$=time after he turns. $3x+6=\frac{192}{7}$=perpendicular, $5y$=base, and $5(\frac{64}{7}-y)$=hypotenuse. Hence, $25y^2+(\frac{192}{7})^2=25(\frac{64}{7}-y)^2$, whence $y=2\frac{162}{175}$ hours.

Excellent solutions of this problem were received from *G. B. M. Zerr, P. S. Berg, J. K. Ellwood, Cooper D. Schmitt, and J. F. W. Scheffer.*

40. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the market-price of $m=3\%$-stock, in order that it may yield $n=3\frac{1}{3}\%$ interest after deducting $d=\$\frac{7}{20}$ from every $S=\$12$.

Solution by the PROPOSER.

According to the conditions of the problem, the deduction from the the par (\$100) value of a share is $100d/S$ dollars, $=\$\frac{35}{12}$; therefore, $100(1-d/S)$ dollars are to yield \$$m$ interest. In order to yield \$$n$ interest,

the market-price must be $P=100\left(\frac{m}{n}\right)\left(1-\frac{d}{S}\right)$dollars, $=\$87\frac{3}{8}$.

COR.—Put $m=n$; then $P=\$97\frac{1}{12}$, which is the correct result of this problem as proposed in the December, '94, MONTHLY.—*F. P. M.*

41. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If I gain \$2 in \$5 by selling a horse for \$150, what per cent. would I gain by selling the horse for \$120?

Solution by P. S. BERG, Apple Creek, Ohio, and the PROPOSER.

Since gaining \$2 in \$5 is gaining 40%, the cost of the horse is $\$107\frac{1}{7}$. Hence the gain required is 12%.

PROBLEMS.

46. Proposed by T. W. PALMER, Professor of Mathematics, University of Alabama.

A borrows \$500.00 from a Building and Loan Association and agrees to pay

\$9.50 per month for 72 months, the first payment to be made at the end of the first month. What rate of interest does A pay? The Association claims to charge only 8 per cent. (the legal rate in Alabama). How can 8 per cent. be figured out on the above.

47. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Mr. Merchant sells 20% above cost, with weights and measures $12\frac{1}{2}$% "short," allows a discount of \$5 on every bill of \$50, and loses 5% of his sales as "bad debts." Find his *rate per cent* of net profit, or net loss; one cent in every dollar of sale proves counterfeit, and collection-charges are $2\frac{1}{2}$%.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

36. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

Resolve $(x^2+y^2)(x^2+z^2)(y^2+z^2)$ into the sum of two squares.

I. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

By Euler's theorem we have

$$(x^2+y^2)(x^2+z^2)=(x^2+yz)^2+(xz\mp xy)^2=A^2+B^2.$$

$$(x^2+y^2)(x^2+z^2)(y^2+z^2)=(A^2+B^2)(y^2+z^2)=(Ay\pm Bz)^2+(Az\mp By)^2$$

$$=\{(x^2\pm yz)y\pm(z\mp y)xz\}^2+\{(x^2\pm yz)z\mp(z\mp y)xy\}^2.$$

$\therefore$ the sum of two squares in four ways.

II. Solution by the PROPOSER.

By determinants we have

$$(x^2+y^2)(x^2+z^2)(y^2+z^2)=\begin{vmatrix} x, & -y \\ y, & x\end{vmatrix}\begin{vmatrix} z, & x \\ -x, & z\end{vmatrix}\begin{vmatrix} y, & z \\ -z, & y\end{vmatrix}$$

$$=\begin{vmatrix} xyz-xy^2-x^2z-yz^2, & -x^2y-xz^2-y^2z+xyz \\ x^2y+xz^2+y^2z-xyz, & xyz-xy^2-x^2z-yz^2\end{vmatrix}$$

$$=(xyz-xy^2-x^2z-yz^2)^2+(x^2y+xz^2+y^2z-xyz)^2.$$

[Other forms can be similarly obtained.—EDITOR].

Also solved by *John Faught, M. A. Gruber, J. Scheffer, and C. D. Schmitt.*

37. Proposed by H. M. CASH. Gibson, Ohio

The area of the segment of a circle=c, and radius=r. Find height of segment.

Solution by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Denoting the arc by 2θ, and the height by h, we have for the area of the segment $r^2\theta-r^2\sin\theta\cos\theta=r^2\theta-\frac{1}{2}r^2\sin2\theta$.

$\therefore\quad 2\theta-\sin2\theta=\frac{2c}{r^2}$, which transcendental equation is to be solved to find θ, then h is found by the relation, $h=2r\sin^2\frac{1}{2}\theta$.

We might express the area also by r and h directly, and we would then have the transcendental equation $\cos^{-1}\frac{r-h}{r}-(r-h)\sqrt{(2rh-h^2)}=\frac{c}{r^2}$; which, however, is too inconvenient for solution.

Also solved by *A. H. Bell*, *J. A. Calderhead*, and *C. D. Schmitt*.

38. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A man sold 2 horses and a mule for \$286.90. On the first horse he gained as much per cent. as the horse cost dollars, and gained $\frac{5}{8}$ as much per cent. on the second horse as the first, and he looses \$9.10 on the mule. His net gain was \$86.90. What was the cost and selling price of each?

Solution by T. W. PALMER, M. A., Professor of Mathematics, University of Alabama.

Let x=cost of 1st horse, y=cost of 2nd horse, and z=cost of mule.

Then $x+y+z=\$286.90-\$86.90=\$200\ldots.(1)$.

$\frac{x^2}{100}+\frac{5xy}{800}=\$86.90+\$9.10=\$96\ldots.(2)$.

$\therefore\quad 8x^2+5xy=76800$, or $y=\frac{8(9600-x^2)}{5x}\ldots.(3)$.

And substituting this in (1), we obtain $z=\frac{3x^2+1000x-76800}{5x}$.

From these two indeterminate equations, we find $x<97$ and $x>65$, and $z>9.10$. An indefinite number of solutions is possible. But in (3), $x=80$ satisfies all the conditions. When $x=80$, $y=64$, and $z=56$.

Also $x+\frac{x^2}{100}=144$, $y+\frac{5xy}{800}=96$, and $z-9.10=46.90$.

$$\therefore\quad \left.\begin{array}{l} x=\$80 \\ y=\$64 \\ z=\$56 \end{array}\right\}=\text{cost};$$

$$\left.\begin{array}{l} x+\frac{x^2}{100}=\$144 \\ y+\frac{5xy}{800}=\$\ 96 \\ z-9.10=\$\ 46.90 \end{array}\right\}=\text{selling price}.$$

Also solved by *J. A. Calderhead*, *J. Scheffer*, *C. D. Schmitt*, and *G. B. M. Zerr*.

A solution of number 35 was received from *Prof. Cooper D. Schmitt*, after last issue of MONTHLY had gone to press.

PROBLEMS.

48. Proposed by SETH PRATT, C. E., Assyria, Michigan.

What is the interest of $500 for 10 years at 10% per annum, when the intervals of compounding are infinitely small?

49. Proposed by P. S. BERG, Apple Creek, Ohio.

A man having lent $6000 at 6% interest, payable quarterly, wishes to receive his interest in equal proportions monthly, and in advance: how much ought he to receive each month?

GEOMETRY.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

36. Proposed by O. W. ANTHONY, Mexico, Missouri.

From two points, one on each of the opposite sides of a parallelogram, lines are drawn to the opposite vertices. Through the points of intersection of these lines a line is drawn. Prove that it divides the parallelogram into two equal parts.

II. Solution by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

The line in question IK will cut the parallelogram into two equal parts if it is proved to pass through O, the intersection of the diagonals. The latter theorem is true for any quadrilateral (Pappus, *Mathematicae Collectiones*, VII, p.139).

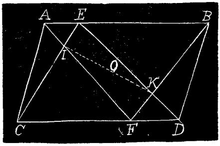


Fig. 1.

If a hexagon $AB'CA'BC'$ has its summits of even order upon one straight line and those of odd order upon another, the three pairs of opposite sides (AB' and $A'B$, BC' and $B'C$, CA' and $C'A$) intersect in three points on a straight line. See Cremona for proof. But even this is only Pascal's theorem for a hexagon inscribed in a conic, when the latter degenerates to a pair of straight lines.

Analytical proof for parallelogram:

Take as axes lines ∥ to AB and BD through O. Let $AB=2b$, $BD=2a$. Call $I(x',y')$ and $K(x_1, y_1)$. Equation to AF is $(x+b)(y'-a)=(y-a)(x'+b)$. Equation to DE is $(x+b)(y'+a)=(y+a)(x'+b)$. Hence the co-ordinates of F are $\left(\frac{2ax'+by'+ab}{a-y'}, -a\right)$; of E, $\left(\frac{2ax'-by'+ab}{a+y'}, a\right)$.

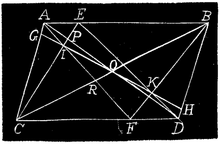
Hence, equation to BF is $a(x-b)(y'-a)=(y-a)(ax'+by')$; equation to ED is $a(x-b)(y'+a)=(y+a)(ax'-by')$.

$\therefore$ the co-ordinates of K are $x_1=\frac{a^2x'(x'+b)}{a^2x'+by'^2}$; $y_1=\frac{a^2y'(x'+b)}{a^2x'+by'^2}$.

$\therefore$ $\frac{x_1}{y_1}=\frac{x'}{y'}$, or IK passes through the origin O.

III. **Solution by J. K. ELLWOOD, A. M., Principal Colfax School, Pittsburg, Pennsylvania.**

Let E and F be the points taken in opposite sides of the parallelogram $ABCD$, GH the line drawn through the points of intersection.



FIRST.—The intersection, O, of the diagonals of $ABCD$ is in the line GH.

The △'s CIF and AIE are similar, as are FKD and EKB, and AOB and COD. Hence

$BO:OC::AB:CD$; or $BO\times CD=OC\times AB$....(1).

$FK:BK::FD:EB$; or $FK\times EB=BK\times FD$....(2).

$CI:EI::FI:AI$; by composition,

$CI:CE::FI:AF$; or $CI\times AF=CE\times FI$....(3).

The sides produced of the △'s CIF, AIE, CIR are cut by the lines DPA, COB, and OA, respectively.

$\therefore$ $CP\times FD\times AI=CD\times AF\times IP$*....(4).

$IR\times AB\times CE=AR\times EB\times CI$....(5).

$CO\times IP\times AR=CP\times AI\times OR$....(6).

Multiplying equations (1),(2),(3),(4),(5),(6) together, we have

$BO\times FK\times IR=BK\times FI\times OR$....(7).

$\therefore$ (By appended proof **) O is in the line $GIKH$.

SECOND.—In the similar △'s AOG and HOD, $AO=OD$.

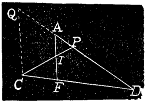
$\therefore$ $HD=AG$. $\therefore$ $GC=HB$, and $AG+HB=DH+CG$.

Area trapezoid $AGHB=\frac{1}{2}(AG+HB)\times$alt.

Area trapezoid $GCDH=\frac{1}{2}(DH+CG)\times$alt.

Then, since $AG+HB=DH+CG$, these areas are equal, and the line GH divides $ABCD$ into two equal parts. *Q. E. D.*

EXPLANATORY PROOF. Let CFI (see fig. 1.) be any $\triangle$, and produce the sides to D, P, A in any *straight line*, as DA.



Through C draw CQ parallel to FA and meeting DA, produced, in Q. The $\triangle$'s DAF and DQC are similar, as are API and QPC.

Hence $FD:AF::CD:CQ$, and

$AI:IP::CQ:CP$.

$\therefore\quad FD\times AI:AF\times IP::CD:CP$.

$\therefore\quad FD\times AI\times CP=AF\times IP\times CD$.

This is equation (4)above; (5) and (6) may be similarly obtained.

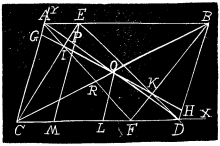
It may be shown in a similar manner that, if a *straight line* cuts two sides and the third side produced of a $\triangle$, the product of any three of the non-adjacent segments (of the sides) is equal to the product of the other three segments. The produced side is one segment, the prolongment another.

*** CONVERSELY. If three points divide the two sides and determine the prolongment of the third side produced so that the product of any three non-adjacent segments shall be equal to the product of the other three, then are these points in the same *straight line*.

Equation (7) is derived from the $\triangle$ BFR. The points K, O are in the *sides*, and I is in RF produced. Since in equation (7) the product of three non-adjacent segments is equal to the product of the other three, the three points I, O, K are in the straight line GH.

IV. **Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Let E, F be the points. I, K the intersection of ED, FB and EC, FA, O the intersection of the diagonals BC, AD. Let $CD=AB=a$, $AC=ME=BD=b$, $CM=c$, $CF=d$, $h=$the perpendicular from A on BD.



Then $CL=\frac{1}{2}a$, $LO=\frac{1}{2}b$,

$\frac{y}{x}=\frac{b}{c}$, equation to CE; $\frac{x}{d}+\frac{y}{b}=1$, equation to AF. $y=\frac{b(a-x)}{a-c}$, equation to DE, $y=\frac{b(x-d)}{a-d}$. equation to BF.

$\left(\frac{cd}{c+d},\frac{bd}{c+d}\right)$ co-ordinates of I, $\left(\frac{a^2-dc}{2a-c-d},\frac{b(a-d)}{2a-c-d}\right)$ co-ordinates of K.

$(2cd-ac-ad)y-(bd-bc)x=bd(c-a)$, the equation to IO.

This line cuts DE at the point $\left(\frac{a^2-dc}{2a-c-d},\frac{b(a-d)}{2a-c-d}\right)$

$\therefore$ IO passes through K, and IO and GH are the same line.

In the triangle GOA and HKD, $AO=OD$, $\angle AOG=\angle DOH$,

$\angle OAG = \angle ODH$. $\therefore$ $GA = DH$. Similarly $CG = BH$.

$\therefore$ $AG + BH = CG + DH$.

$\therefore$ $\frac{1}{2}h(AG + BH) = \frac{1}{2}h(CG + DH)$.

$\therefore$ Area $AGHD$ = area $CGHD$.

Good solutions of this problem were received from *Professors Wm. Symmonds, and Cooper D. Schmitt.*

This problem has proved to be a very interesting one and for that reason we have given it extra space. The proposition to which Prof. Ellwood has given a proof explanatory to the proposition under consideration is known as the proposition of *Menelaus*. See *Halsted's Elementary Synthetic Geometry*, p. 117. Editor.

PROBLEMS.

42. Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.

If the bisectors of two angles of a triangle are equal the triangle is isosceles.

[The term *bisector* in this theorem means the line which divides an angle into two equal parts and terminates in the opposite side.]

43. Proposed by J. F. W. SCHEFFER, Hagerstown, Maryland.

The consecutive sides of a quadrilateral are a, b, c, d. Supposing its diagonals to be equal, find them and also the area of the quadrilateral.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

27. Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

A runs around the circumference of a circular field with velocity m feet; B starts from the centre with velocity $n > m$ feet to catch A. The straight line joining their positions always passes through the centre. Find the equation to the curve described by B, the distance he runs and the time occupied.

I. Solution by A. H. HOLMES, Brunswick, Maine, H. W. DRAUGHON, Olio, Mississippi, and the PROPOSER.

Let A be the point of starting of the pursued. P, B, the position of the pursuer and pursued at any time.

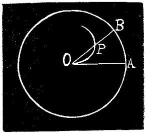
Let $OA = a$, $OP = r$, $\angle BOA = \theta$, $\frac{m}{n} = u$, arc $OP = s$.

Then $us=a\theta \dots (1)$. $\therefore$ $s=\frac{a\theta}{u}=\frac{an\theta}{m}$ is the intrinsic equation to the curve.

From (1) $\frac{ds}{d\theta}=\frac{a}{u}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$.

$$\therefore\ r^2+\left(\frac{dr}{d\theta}\right)^2=\frac{a^2}{u^2} \text{ and } d\theta=\frac{dr}{\sqrt{\left(\frac{a^2}{u^2}-r^2\right)}}.$$

$\therefore$ $\theta=\sin^{-1}\frac{ur}{a}\dots(2)$.



$\therefore$ $r=\frac{a}{u}\sin\theta=\frac{an}{m}\sin\theta$, is the polar equation and $m(x^2+y^2)=any$, is the rectangular equation.

The value of θ from (2) in (1) gives

$$s=\frac{a}{u}\sin^{-1}\frac{ur}{a},$$

the length for any value of r. When $r=a$,

$$s=\frac{a}{u}\sin^{-1}u=\frac{an}{m}\sin^{-1}\frac{m}{n}=\text{distance run.}$$

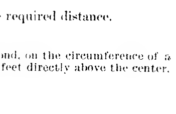
$$t=\text{time}=\frac{s}{n}=\frac{a}{m}\sin^{-1}\frac{m}{n}.$$

Also solved by *Professors O. W. Anthony, J. Scheffer, and William Symmonds.*

28. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

How far from the stage must Miss Love sit in order that she *may see to best advantage* Mr. Rich deliver the valedictory oration?

Solution by O. W. ANTHONY, Missouri Military Academy, Mexico, Missouri, and the PROPOSER.



Let E represent the position of Miss Love's eyes; DB the stage from which Mr. Rich orates; $AB,=m$ feet, the height of the stage above Miss Love's eyes; $BC,=n$ feet, the height of Mr. Rich; and $AE,=x$ feet, the required distance. In order that Miss Love may *see* Mr. Rich *to best advantage*, the angle BEC must be a maximum, that is,

$$U=\tan^{-1}\left(\frac{m+n}{x}\right)-\tan^{-1}\left(\frac{m}{x}\right)=\text{a Maximum.}$$

$$\therefore\ \frac{dU}{dx}=\frac{m}{x^2+m^2}-\frac{m+n}{x^2+(m+n)^2}=0\dots(1).$$

Whence $x=\sqrt{[m(m+n)]}$ feet, which is the required distance.

29. Proposed by CHARLES E. MYERS, Canton, Ohio

A hen running at the rate of $n=2$ feet per second, on the circumference of a circle, radius $r=50$ feet, is observed by a hawk $a=600$ feet directly above the center.

The hawk at once starts in pursuit, flying at the rate of $m=5$ feet per second and keeping always in a straight line with the starting point and the hen.

Determine the path followed and the distance the hawk will fly before catching the hen.

Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let the origin be at the vertex of the cone around which the path of the hawk winds. σ=length of the hawk's path, s=the length of the projection of this path on the plane (x,y), ρ=radius vector of this projection, $\frac{n}{m}=u$, $\frac{r}{a}=c$. Then $x^2+y^2=c^2z^2$ is the equation of the cone, also $n\sigma=r\theta$, where θ is the angle subtended by the hen's path at the centre of the circle.

$$\therefore\ d\sigma=\frac{r}{u}\,d\theta=\sqrt{dx^2+dy^2+dz^2}=\sqrt{ds^2+dz^2}$$

$$=\left\{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}}d\theta,\ \text{but}\ \rho^2=x^2+y^2=c^2z^2.\quad\therefore\ \rho=cz.\quad\therefore\ dz=\frac{d\rho}{c}.$$

$$\therefore\ \frac{d\sigma}{d\theta}=\frac{r}{u}=\left\{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2+\frac{1}{c^2}\left(\frac{d\rho}{d\theta}\right)^2\right\}^{\frac{1}{2}}.$$

$$\therefore\ \frac{d\theta}{d\rho}=\frac{\sqrt{c^2+1}}{c}\cdot\frac{1}{\left\{\frac{r^2}{u^2}-\rho^2\right\}^{\frac{1}{2}}}.\quad \theta=\frac{\sqrt{c^2+1}}{c}\sin^{-1}\frac{u\rho}{r}.$$

$$\therefore\ \theta=\frac{\sqrt{r^2+a^2}}{r}\sin^{-1}\left(\frac{n\rho}{mr}\right);\ \rho=\frac{mr}{n}\sin\frac{r\theta}{\sqrt{r^2+a^2}}.$$

$$z=\rho/c=\frac{am}{n}\sin\frac{r\theta}{\sqrt{r^2+a^2}}=1500\sin.\frac{\theta}{\sqrt{145}}.$$

$$x=\rho\cos\theta=\frac{mr}{n}\sin\frac{r\theta}{\sqrt{r^2+a^2}}\cos\theta=125\sin\frac{\theta}{\sqrt{145}}\cos\theta.$$

$$y=\rho\sin\theta=\frac{mr}{n}\sin\frac{r\theta}{\sqrt{r^2+a^2}}\sin\theta=125\sin\frac{\theta}{\sqrt{145}}\sin\theta.$$

These values of x, y, z determine the hawk's path.

Also $n\sigma=r\theta$. $\therefore\ \sigma=\frac{mr}{n}\left[\frac{\sqrt{r^2+a^2}}{r}\sin^{-1}\left(\frac{n\rho}{mr}\right)\right]_0^r=\frac{m\sqrt{r^2+a^2}}{n}\sin^{-1}\left(\frac{n}{m}\right)$.

$\therefore\ \sigma=125\sqrt{145}\sin^{-1}\left(\frac{2}{5}\right)=619.406$ feet approximately, the distance the hawk flies before catching the hen.

Solved in a similar manner by *Professor J. F. W. Scheffer.*

PROBLEMS.

38. **Proposed by L. B. FILLMAN, St. Petersburg, Pennsylvania.**

The diameter of the circular base of a dome is $10=a$ feet, which is also the distance from any point on the circumference of the base to any point on the opposite side of the dome from base to apex. Find volume of dome. [See Prob. 21. EDITOR.]

39. **Proposed by J. C. GREGG, Brazil, Indiana.**

Show that the curve

$$x=9a\sin\theta-4a\sin^2\theta$$
$$y=-3a\cos\theta+4a\cos^2\theta$$

is symmetrical to the axes, and has double points and cusps: find the lengths of the arcs, (a) between the double points, (b) between a double point and a cusp, (c) and the arc connecting two cusps and not passing through a double point. [*Rice and Johnson's Integ. Cal.* (abridged), p. 176.]

MECHANICS.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

15. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Show that the *eastward deviation* of bodies falling from a great height is

$$E_d=\frac{4\pi t(H-\frac{1}{2}\triangle)\cos\phi}{3T}.$$

II. **Solution by Professor G. B. M. ZERR, A. M., Principal of High School Staunton, Virginia.**

Let OZ be the axis of z, OX, the axis x, OY, the axis of y.

Let ϕ=the latitude of O,

β=angular velocity of the earth around DO,

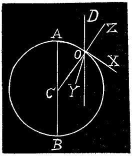
$\triangle$=excess of descent in vacuo over that of air,

T=time in seconds of siderial day,

H=height of body above the earth.

Also let OX be tangent to the meridian and OY perpendicular to it, and their positive directions respectively south and west. The velocity of the body eastward at the moment it is dropped from $z_1=\beta H\cos\phi$. Now if gravity did not alter its direction, owing to the rotation of the earth, the body would describe a parabola and the easterly deviation would be $(\beta H\cos\phi)t$, where t=time of falling. But the rotation β about OD is equivalent to $\beta\sin\phi$ about

OZ, and $\beta\cos\phi$ about OX. The former does not alter the position of OC, while the latter turns OC in a time t through an angle $\beta\cos\phi t$. Hence the body is acted upon by a westerly component, due to the change of direction of gravity, $=g\sin(\beta\cos\phi t)=g\beta\cos\phi t$, since βt is small. Now let y be the distance the body is in space from the plane XZ at the moment the body begins to fall, and then the equation of motion of the body in space is $\frac{d^2y}{dt^2}=g\beta t\cos\phi$. Integrating this, and remembering that, as explained above, $\frac{dy}{dt}=-\beta H\cos\phi$, we get $y=-\beta Ht\cos\phi+\frac{1}{6}g\beta t^3\cos\phi$. $\therefore$ $E_d=\beta t\cos\phi(H-\frac{1}{6}gt^2)$. But $\frac{1}{2}gt^2=H+\triangle$ and the centrifugal force at the equator $=\beta^2r=\frac{4\pi^2r}{T^2}$, where r= radius of the earth.



$\therefore$ $\beta^2=\frac{4\pi^2}{T^2}$ and $\beta=\frac{2\pi}{T}$. Substituting we get

$$E_d=\frac{2\pi t\cos\phi}{T}\left\{H-\tfrac{1}{3}(H+\triangle)\right\}=\frac{2\pi t\cos\phi}{T}(\tfrac{2}{3}H-\tfrac{1}{3}\triangle)$$

$$=\frac{2\pi t\cos\phi}{3T}(2H-\triangle)=\frac{4\pi t(H-\frac{1}{2}\triangle)\cos\phi}{3T}.$$

[NOTE.—No solution has yet been received to problem 17. EDITOR.]

PROBLEMS.

24. Proposed by Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

A sphere whose center of gravity does not coincide with its geometrical center is placed on a rough inclined plane. State under what circumstances the sphere will slide without rolling, roll without sliding, and neither roll nor slide.

25. Proposed by Professor GEORGE LILLEY, LL. D., Ex-President of Washington State Agricultural College and School of Science, Portland, Oregon.

It is known that if the velocity of a certain freight train is 30 miles an hour it can be brought to a stand still in a distance of 500 feet by setting the brakes. It was stopped in 1200 feet by setting the brakes. Find its velocity, the forces exerted by the brakes being the same in each case.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

18. Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Decompose into the sum of two squares the number $17^3.73^5$.

I. Solution by the PROPOSER.

$$(a^2+b^2)(a^2+b^2)=(a^2\pm b^2)^2+(ab\mp ab)^2=A^2+B^2.$$

$$(a^2+b^2)^3=(A^2+B^2)(a^2+b^2)=(Aa\pm Bb)^2+(Ab\mp Ba)^2.$$

$$\therefore\quad (a^2+b^2)^3=\{a(a^2-3b^2)\}^2+\{b(3a^2-b^2)\}^2,$$
$$=\{a(a^2+b^2)\}^2+\{b(a^2+b^2)\}^2.$$

Similarly $(c^2+d^2)^5=\{c(c^4-10c^2d^2+5d^4)\}^2+\{d(5c^4-10c^2d^2+d^4)\}^2$.

$$=\{c(c^2+d^2)^2\}^2+\{d(c^2+d^2)^2\}^2,$$
$$=\{c(c^2+d^2)(3d^2-c^2)\}^2+d(c^2+d^2)(3c^2-d^2)\}^2.$$

Let $a=4$, $b=1$, $c=8$, $d=3$. $\therefore\quad 17^3=52^2+47^2=68^2+17^2$.

$73^5=10072^2+44403^2=42632^2+15987^2=21608^2+40077^2$.

$$\therefore\quad 17^3.73^5=3092572^2+788035^2=2357900^2+2150653^2$$
$$=362372^2+3170755^2=1811860^2+2627197^2$$
$$=3190628^2+69955^2=2848180^2+1439747^2$$
$$=3099580^2+560003^2=1068428^2+3007235^2$$
$$=2835028^2+1465475^2=1172380^2+2968253^2$$
$$=2782340^2+1565197^2=1835572^2+2612685^2.$$

$\therefore$ the sum of two squares in twelve ways.

II. Solution by H. W. DRAUGHON, Olio, Mississippi, and O W. ANTHONY, Missouri Military Academy, Mexico, Missouri.

Since $17^3\times73^5=17^2\times73^4\times17\times73=17^2\times73^4\times1241$.

And $1241=35^2+4^2=29^2+20^2$. Therefore,

$$17^3\times73^5=(17\times73^2\times35)^2+(17\times73^2\times4)^2=(17\times73^2\times29)^2+(17\times73^2\times20)^2.$$

III. Solution by COOPER D. SCHMITT, A M, Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Solution by determenants,

$$17=\begin{vmatrix}4 & -1\\ 1 & 4\end{vmatrix},\qquad 73=\begin{vmatrix}8 & -3\\ 3 & 8\end{vmatrix}\qquad 17^2=\begin{vmatrix}17 & 0\\ 0 & 17\end{vmatrix}\qquad 73^4=\begin{vmatrix}5329 & 0\\ 0 & 5329\end{vmatrix}$$

$$17^3=\begin{vmatrix}17 & 0\\ 0 & 17\end{vmatrix}\times\begin{vmatrix}4 & -1\\ 1 & 4\end{vmatrix}=\begin{vmatrix}68 & 17\\ -17 & 68\end{vmatrix}$$

$$17^3\times73=\begin{vmatrix}68 & 17\\ -17 & 68\end{vmatrix}\times\begin{vmatrix}8 & -3\\ 3 & 8\end{vmatrix}=\begin{vmatrix}344-51 & 204+136\\ -136-204 & -51+344\end{vmatrix}=\begin{vmatrix}493 & 340\\ -340 & 493\end{vmatrix}.$$

$$\text{Then } 17^3\times73^5=\begin{vmatrix}493 & 340\\ -340 & 493\end{vmatrix}\times\begin{vmatrix}5329 & 0\\ 0 & 5329\end{vmatrix}=(493^2+340^2)(5329)^2$$

$=(73)^2(493)^2+(73)^2(340)^2$, which is one set of answers.

Also solved by *R. J. Adcock*, and *M. A. Gruber*.

19. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C

Find three positive integer numbers whose sum is a cube, and, also, the sum of any two diminished by the third a cube.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let x, y, and z=the three positive integers.

$$\text{Then } \begin{aligned} x+y+z&=a^3\\ x+y-z&=b^3\\ x+z-y&=c^3\\ y+z-x&=d^3\end{aligned}$$

Whence $x+y+z=b^3+c^3+d^3=a^3$;

$$x=\frac{b^3+c^3}{2}\ ;\ y=\frac{b^3+d^3}{2}\ ;\ z=\frac{c^3+d^3}{2}.$$

This is a problem in which the sum of three cubes=a cube. Take $3^3+4^3+5^3=6^3$. But as the numbers are to be integers, we multiply by 2, and obtain $6^3+8^3+10^3=12^3$. $\therefore\ x=\dfrac{6^3+8^3}{2}=364$; $y=\frac{1}{2}(6^3+10^3)=608$; and $z=\frac{1}{2}(8^3+10^3)=756$. The number of answers is infinite.

II. Solution by R J. ADCOCK, Larchland, Warren County, Illinois.

Let x, y, z, be the three numbers; then $x+y+z=u^3$, $x+y-z=r^3$, $x+z-y=r^3$, $z+y-x=s^3$, by conditions.

Wherefore $x+y+l=u^3=r^3+p^3+s^3$, $x=\dfrac{r^3+p^3}{2}$, $y=\frac{1}{2}(r^3+s^3)$, $z=\frac{1}{2}(r^3+s^3)$. The most general equation yet obtained by me for the sum of three cubes=a cube, is found from,

$[(ax^3+dy^3)x]^3+[(bx^3+hy^3)y]^3+[(cx^3-hy^3)y]^3=[(ax^3+gy^3)x]^3$, by expanding, equating coefficients of similar terms with respect to x and y, eliminating d, g, and h, giving the identical equation,

$$[9a^3bx^3y+(b^2-bc+c^2)^2y^4]^3+[9a^3cx^3y-(b^2-bc+c^2)^2y^4]^3$$
$$+[9a^4x^4-3(b^2-bc+c^2)caxy^3]^3=[9a^4x^3+3ab(b^2-bc+c^2)xy^3]^3.$$

By numbers for the letters in the above, some of the resulting

equations are $3^3+4^3+5^3=6^3$, $1^3+6^3+8^3=9^3$, $3^3+10^3+18^3=19^3$,

$$7^3+14^3+17^3=20^3,\ 4^3+17^3+22^3=25^3,\ 11^3+15^3+27^3=29^3.$$

Then the three positive integer numbers are $x=\dfrac{11^3+15^3}{2}$, $y=\dfrac{11^3+27^3}{2}$, $z=\dfrac{15^3+27^3}{2}$. Also x, y, z may be found from any equation, including the algebraicsum, for the sum of three cubes = a cube, by first multiplying each cube by 2^3.

Also solved by *O. W. Anthony, H. W. Draughon, C. D. Schmitt, and G. B. M. Zerr.*

PROBLEMS.

27. **Proposed by J. W. NICHOLSON, LL. D., President and Professor of Mathematics, Louisiana State University and A. and M. College, Baton Rouge, Louisiana.**

Required a formula for finding five integers the sum of whose cubes is a cube.

28. **Proposed by DAVID E. SMITH, Ph. D., Professor of Mathematics, Michigan State Normal School, Ypsilanti, Michigan.**

Decompose the product 97.673.257 into the sum of two fourth powers.

AVERAGE AND PROBABILITY.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

13. **Proposed by I. L. BEVERAGE, Monterey, Virginia.**

Find the mean values of the roots of the quadratic $x^2-ax+b=0$, the roots being known to be real, but b being unknown and positive.

Solution by P. S. BERG, Apple Creek, Ohio, and JOHN DOLMAN, Jr., Counsellor-at-law, Philadelphia, Penn., and J. M. COLAW, A. M., Principal of High School, Monterey, Virginia.

Solving the given equation, $x=\frac{1}{2}a\pm\sqrt{(\frac{1}{4}a^2-b)}$.

Therefore, if b be positive and x real, b cannot exceed $\frac{1}{4}a^2$. If β be the smaller of the two roots, its mean value, therefore, is

$$(1\div\tfrac{1}{4}a^2)\int_0^{\frac{1}{4}a^2}\beta db=\frac{4}{a^2}\int_0^{\frac{1}{4}a^2}(\tfrac{1}{2}a+\sqrt{(\tfrac{1}{4}a^2-b)}db=\frac{4}{a^2}[\tfrac{1}{2}ab]_0^{\frac{1}{4}a^2}+\frac{4}{a^2}[\tfrac{1}{12}\sqrt{(a^2-4b)^3}]_0^{\frac{1}{4}a^2}$$

$$-\frac{a}{3}=\frac{a}{2}-\frac{a}{3}=\frac{1}{6}a.$$

The mean value of the larger root is, therefore, $\frac{5}{6}a$.

Also solved in a similar manner by *Professors Matz, Zerr, and Draughon.*

14. Proposed by CHARLES E. MYERS, Canton, Ohio.

$\frac{1}{3}$ of all the melons in a patch are not ripe, and $\frac{1}{4}$ of all the melons in the same patch are rotten, the remainder being good. If a man enters the patch on a dark night and takes a melon at random, what is the probability that he will get a good one?

Solution by H. W. DRAUGHON, Olio, Mississippi, and G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $12n$=the whole number of melons in the patch. Then $4n$ are not ripe and $3n$ are rotten. The $3n$ rotten melons may be included in the $4n$ not ripe melons in which case there would be $8n$ good melons, or the $3n$ rotten may not be included in the $4n$ not ripe melons in which case there would be $12n-(3n+4n)=5n$ good melons.

$\therefore$ there cannot be less than $5n$ nor more than $8n$ good melons.

$$\therefore\quad \text{the chance of a good one}=\tfrac{1}{2}\left(\frac{5n+8n}{12n}\right)=\tfrac{13}{24}.$$

$$\text{The chance of a not ripe one}=\tfrac{1}{2}\left(\frac{n+4n}{12n}\right)=\tfrac{5}{24}.$$

$$\text{The chance of a rotten one}=\tfrac{1}{2}\left(\frac{0+3n}{12n}\right)=\tfrac{1}{8}.$$

$$\text{The chance of a not ripe and rotten one}=\tfrac{1}{2}\left(\frac{0+3n}{12n}\right)=\tfrac{1}{8}.$$

$\therefore\quad \frac{13}{24}+\frac{5}{24}+\frac{1}{8}+\frac{1}{8}=1$ as it should be.

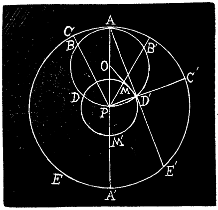
Solutions o this problem were received from *P. S. Berg, F. P. Matz, J. M. Colaw.*

15. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Todhunter proposes: "From a point in the circumference of a circular field a projectile is thrown at random with a given velocity, which is such that the diameter of the field is equal to the greatest range of the projectile: prove the chance of its falling within the field, is $C=2^{-1}-2\pi^{-1}(\sqrt{2}-1),=.236+$." Is this result perfectly correct as to fact?

First Solution by the PROPOSER.

Let P be the point from which the projectile is thrown, $AP=2a$, and $\angle APB=\theta$. Now, if ϕ=the angle of elevation at which the projectile is thrown, and C'=the chance for any given value of θ; then, evidently, the required chance becomes



$C=\int_0^{\frac{1}{2}\pi} C' \, d\theta \div \int_0^{\pi} d\theta = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} C' \, d\theta \ldots (1).$

Since the range is

$PB = 2a \sin 2\phi = 2a \cos\theta,$

$\sin\phi = \frac{1}{2}[\sqrt{(1+\cos\theta)} + \sqrt{(1-\cos\theta)}], = R_1,$

and $\sin\phi = \frac{1}{2}[\sqrt{(1+\cos\theta)} - \sqrt{(1-\cos\theta)}],$ $= R_2.$

For all values of $\sin\phi$ *less* than R_2 and *greater* than R_1, the projectile will fall into the field. The whole number of different directions of projection is proportional to the surface S_1 of the hemisphere center of which is P and radius $\frac{1}{2}PB = a\cos\theta$; and this surface is

$$S_1 = 2\pi a^2 \cos^2\theta \ldots (2).$$

The whole number of different directions of projection producing a range *greater* than PB is proportional to the surface S_2 of the zone included between two horizontal planes at the distances $R_1 a\cos\theta$ and $R_2 a\cos\theta$ from the center of the base of the hemisphere; and this surface is

$$S_2 = a\cos\theta\sqrt{(1-\cos\theta)} \times 2\pi a \cos\theta = 2\pi a^2 \cos^2\theta \sqrt{(1-\cos\theta)} \ldots (3).$$

That is, the whole number of different directions of projection giving a range *less* than PB is proportional to $S_1 - S_2$; and, therefore, we have the chance for any given value of θ

$$C' = \frac{S_1 - S_2}{S_1} = 1 - \frac{S_2}{S_1} = 1 - \sqrt{(1-\cos\theta)} = 1 - \sqrt{(2)}\sin\tfrac{1}{2}\theta \ldots (4).$$

$$\therefore \quad C = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} C' \, d\theta = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} [1 - \sqrt{(2)}\sin\tfrac{1}{2}\theta] d\theta = \frac{1}{2} - 2\left(\frac{\sqrt{2}-1}{\pi}\right)$$

$$= 2^{-1} - 2\pi^{-1}(\sqrt{2}-1), = .236+,$$ which is Todhunter's result.

Second Solution.

Obviously the number of *favorable* chances is represented by the area of the circle $ABDPD'B'A - O$, and the total number of chances is represented by the area of the circle $ACEA'E'C'A - P$. Therefore, the required chance is $C = \frac{1}{4}, = .25$.

Third Solution.

For any range PM the number of *favorable* chances is represented by the area $\mathbf{A}'$ of the double segment $DMD'P$, and the total number of chances is represented by the area of the circle $DM'D'M - P$. Let $\angle PAD' = \omega$; then $\angle POD' = 2\omega$, $\angle MPD' = (90 - \omega)$, $PM = PD' = 2a\sin\omega$, and $\mathbf{A}' = 2(\text{Sector } MPD' + \text{Sector } POD' - \text{Triangle } OD'P) = 2a^2[\pi\sin^2\omega$ $-(\omega + \sin\omega\cos\omega - 2\omega\cos^2\omega)]$. Hence the required chance becomes

$$C = 2a^2\int_0^{\frac{1}{2}\pi} \mathbf{A}' d\omega \div 4\pi a^2 \int_0^{\frac{1}{2}\pi} \sin^2\omega \, d\omega = \frac{1}{2} - \frac{2}{\pi^2}, = .297+.$$

Fourth Solution.

From A draw at random the chord AE', put $\angle A'AE'=\Psi$: then $AD'=2a\cos\Psi=r_1$, and $AE'=4a\cos\Psi=r_2$. For any value of Ψ the number of *favorable* chances is represented by the *sectoral* surface PAD', and the total number of chances by the sectoral surface $A'AE'$. The chance in consideration, therefore, becomes

$$C=2\int_0^{\frac{1}{2}\pi}\int_0^{r_1}d\Psi rdr\div 2\int_0^{\frac{1}{2}\pi}\int_0^{r_2}d\Psi rdr=\frac{1}{4},=.25.$$

Fifth Solution.

Put $\angle PAD'=\omega$, then $\angle APD'=(\frac{1}{2}\pi-\omega)$. Therefore, for any range PD' the projectiles falling on the circular arc DMD' are within the field. Consequently the required chance becomes

$$C=4a\int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}\pi-\omega)\sin\omega d\omega\div 4\pi a\int_0^{\frac{1}{2}\pi}\sin\omega d\omega=\frac{1}{2}-\frac{1}{\pi},=.182.$$

Sixth Solution.

The number of *favorable* chances is proportional to $2\angle MPD'=2(\frac{1}{2}\pi-\Psi)$, and the total number of chances is proportional to $2(\pi)$. Hence the required chance becomes

$$C=2\int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}\pi-\Psi)d\Psi\div 2\pi\int_0^{\frac{1}{2}\pi}d\Psi=\tfrac{1}{4},=.25.$$

NOTE—Since the projectiles are *thrown* at random, they should *fall* at random; and, therefore, the required chance should be $C=\frac{1}{4},=.25$. To interpret *this* result is apparently easy enough; but to interpret Todhunter's result, or the results $C=.297+$ and $C=.182$, is not so easy. In fact, the interpretation of these three results becomes all the more remarkable when we note that their average value $C_A=.238+$, which average value differs but slightly from Todhunter's result.

This problem was also solved by G. B. M. Zerr. J. M. Colaw, and John Dolman, Jr., their result agreeing with that given by Todhunter and the first solution of Professor Matz. Professor Zerr says this result is perfectly true as to mathematical fact. We published all of Professor Matz's solutions for comparison.—EDITOR.

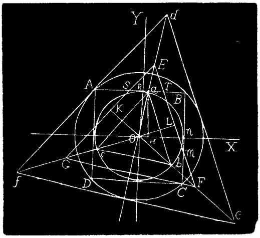
18. Proposed by B. F. FINKEL, A. M., Professor of Mathematics in Kidder Institute, Kidder, Missouri.

What is the average volume common to a cube and a rectangular solid one inch square, the axis of rectangular solid being equal to and coinciding with the diagonal of the cube?

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $ABCD$ be a section of the rectangular solid, EFG a section of

the cube both projected upon the same plane, perpendicular to the diagonal of the cube. The triangle varies in size from *abc*, the inscribed triangle of the inscribed circle of the square, to *def* the circumscribed triangle of the circum-circle of the square. Before the triangle is equal in area to *abc*, all the cube is common to both, after the triangle is greater in area than *def*, all the rectangular solid is common to both.



Let a=edge of cube, c= side of square $ABCD$. Also suppose $c<\frac{a}{\sqrt{3}}$; for if $c>\frac{a}{\sqrt{3}}$ the section of the cube will be both a triangle and a hexagon.

Let x=side of triangle, p=perpendicular from corner of cube to triangle EFG. Then $p=\frac{x}{\sqrt{6}}$ when $x=ab=\frac{c\sqrt{3}}{2}$, $p=\frac{c}{2\sqrt{2}}$, area $abc=\frac{3\sqrt{3}c^2}{16}$

Volume of pyramid common to both $=\frac{1}{3}p\times abc=\frac{c^3\sqrt{3}}{32\sqrt{2}}$.

When $x=de=c\sqrt{6}$, $p=c$.

$\therefore$ volume rectangular solid common to both$=c^2(\frac{1}{2}a\sqrt{3}-c)=\frac{1}{2}c^2(a\sqrt{3}-2c)$.

$\therefore$ the constant volume common to both solids is double the two volumes just found as we have considered but half the cube.

$$\therefore\quad V_1=2\left\{\frac{c^3\sqrt{3}}{32\sqrt{2}}+\tfrac{1}{2}c^2(a\sqrt{3}-2c)\right\}=\frac{c^2}{16\sqrt{2}}\left\{c\sqrt{3}+16a\sqrt{6}-32c\sqrt{2}\right\}.$$

To find the average volume common to both, that varies, let OX, OY be the axis of reference, $\angle EOY=\theta$.

This volume has sections ranging from the triangle through the quadrilateral, pentagon, hexagon, heptagon, and back through its variations to the square. To find this volume I shall divide its altitude into three equal parts and find the average area of sections passing through these points of division, and then apply the formula of approximate cubature.

For the first point of division $x=\frac{c\sqrt{3}}{3}(\sqrt{2}+1)$.

For the second point of division $x=\frac{c\sqrt{3}}{6}(4\sqrt{2}+1)$.

$OL=OK=\frac{x\sqrt{3}}{6}$, $EH=\frac{x\sqrt{3}}{3}\cos\theta$.

Equation to EF is $x_1\cos\left(\frac{\pi}{6}-\theta\right)+y_1\sin\left(\frac{\pi}{6}-\theta\right)=\frac{x\sqrt{3}}{6}$.

Equation to EG is $x_1\cos\left(\frac{5\pi}{6}-\theta\right)+y_1\sin\left(\frac{5\pi}{6}-\theta\right)=\frac{x\sqrt{3}}{6}$.

$RT=\frac{2x\sqrt{3}-3c\cos\theta+3c\sqrt{(3)}\sin\theta}{6[\sqrt{(3)}\cos\theta+\sin\theta]}$, $RS=\frac{2x\sqrt{3}-3c\cos\theta-3c\sqrt{(3)}\sin\theta}{6[\sin\theta-\sqrt{(3)}\cos\theta]}$.

$$nm=\frac{2x\sqrt{3}-3c\sqrt{(3)}\cos\theta-3c\sin\theta}{6[\cos\theta-\sqrt{(3)}\sin\theta]}.$$

When $x=\frac{c\sqrt{3}}{3}(\sqrt{2}+1)$ the triangle EFG is greater than the inscribed triangle of circum-circle of the square $ABCD$. Hence three times the average area of EST subtracted from the area EFG gives the average area of the section required. Call this area A_2. From $\theta=0$ to $\theta=\theta_1$, area $EST=\frac{1}{2}(RT-RS)(EH-\frac{1}{2}c)$; from $\theta=\theta_1$ to $\theta=\frac{\pi}{4}$, the area $=EST+TBM$ $=\frac{1}{2}(RT-RS)(EH-\frac{1}{2}c)+\frac{1}{2}(RT-\frac{1}{2}c)(\frac{1}{2}c-nm)$. θ_1 is determined from the equation $\frac{2x\sqrt{3}-3c\cos\theta+3c\sqrt{(3)}\sin\theta}{6[\sin\theta+\sqrt{(3)}\cos\theta]}=\frac{1}{2}c$.

When $x=\frac{c\sqrt{3}}{6}(4\sqrt{2}+1)$, the triangle EFG is greater than the circumscribed triangle of the in-circle of the square $ABCD$. Hence, three times the average area of the triangle TBM subtracted from the area of the square $ABCD$ gives the average area of the section required. Call this area A_3.

Area $TBM=\frac{1}{2}(\frac{1}{2}c-nm)(\frac{1}{2}c-RT)$. Limits of θ are, $\theta=\theta_2$ to $\theta=\frac{\pi}{4}$, where θ_2 is found from the equation, $\frac{2x\sqrt{3}-3c\sqrt{(3)}\cos\theta-3c\sin\theta}{6[\cos\theta-\sqrt{(3)}\sin\theta]}=\frac{1}{2}c$.

$$\therefore\quad V_2=2x\frac{h}{8}(A_1+3A_2+3A_3+A_4),$$

where $h=c-\frac{c}{2\sqrt{2}}=\frac{c}{2\sqrt{2}}(2\sqrt{2}-1)$. $A_1=$ area $abc=\frac{3\sqrt{(3)}c^2}{16}$. $A_4=$ area $ABCD=c^2$. $\therefore\quad V_2=\frac{3c}{8\sqrt{2}}(2\sqrt{2}-1)(A_2+A_3)+\frac{c}{8\sqrt{2}}(2\sqrt{2}-1)\left(\frac{3\sqrt{(3)}c^2}{16}+c^2\right)=\frac{3c}{8\sqrt{2}}(2\sqrt{2}-1)(A_2+A_3)+\frac{c^3}{128\sqrt{2}}(6\sqrt{6}+32\sqrt{2}-3\sqrt{3}-16)$.

$$V=V_1+V_2=\frac{3c}{8\sqrt{2}}(2\sqrt{2}-1)(A_2+A_3)+\frac{c^3}{128\sqrt{2}}$$

$(6\sqrt{6}+5\sqrt{3}-224\sqrt{2}-16)+ac^2\sqrt{3}$, where A_2 and A_3 are found as indicated above.

[NOTE.—No other solution of this problem was received. Professor Zerr worked on this problem during the hot days of last August. He said the temperature was too high for him to make a complete solution. The problem is more difficult than we supposed. After trying to effect a solution the true character of the problem was revealed us. EDITOR.]

PROBLEMS.

26. **Proposed by J. W. WATSON, Middle Creek, Ohio.**

Find the average area of all right angled triangles having a given hypotenuse.

27. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Find the mean area of the *dodecagonal surface* formed by joining in order the points taken at random, one in each sector of a regular dodecagon.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

12. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

If the measures of curvature and tortuosity of a curve be constant at every point of a curve, the curve will be a helix traced on a cylinder.

Solution by the PROPOSER.

A helix, inclination ω, traced on a right circular cylinder, radius r, is an unicursal curve, and may be defined by the equations $x=r\cos\theta\ldots.(1)$, $y=r\sin\theta\ldots.(2)$, and $z=r\theta\tan\omega\ldots.(3)$, in which θ is the angle through which the generating line has resolved when the point has moved through a space z on the generating line. From (1), (2), and (3), we have respectively

$$\frac{dx}{d\theta}=-r\sin\theta;\ d\left(\frac{dx}{d\theta}\right)=\frac{d^2x}{d\theta^2}=-r\cos\theta\ldots.(4),$$

$$\frac{dy}{d\theta}=r\cos\theta;\ d\left(\frac{dy}{d\theta}\right)=\frac{d^2y}{d\theta^2}=-r\sin\theta\ldots.(5),$$

$$\frac{dz}{d\theta}=r\tan\omega;\ d\left(\frac{dz}{d\theta}\right)=\frac{d^2z}{d\theta^2}=0\ldots.(6).$$

$$\therefore\ \frac{ds}{d\theta}=\sqrt{\left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2\right]}=\frac{r}{\cos\omega}\ldots.(7).$$

Dividing (4), (5), and (6), by the square of (7),

$$\frac{d^2x}{ds^2}=\frac{-\cos\theta\cos^2\omega}{r}\ldots.(8),\ \frac{d^2y}{ds^2}=\frac{-\sin\theta\cos^2\omega}{r}\ldots.(9),\text{ and }\frac{d^2z}{ds^2}=0\ldots.(10).$$

Since the reciprocal of the radius of curvature is the *measure of the curvature* at any point of a tortuous curve, we have

$$\frac{1}{\rho}=\sqrt{\left[\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right]}=\frac{\cos^2\omega}{r}\ldots\ldots(11),$$

which is necessarily a *constant* quantity for every point of the curve.

The formula for the *measure of the tortuosity* at any point of a tortuous curve is, regarding τ as the radius of torsion,

$$\frac{1}{\tau}=\sqrt{\left[\left(\frac{d\lambda}{ds}\right)^2+\left(\frac{d\mu}{ds}\right)^2+\left(\frac{d\nu}{ds}\right)^2\right]}\ldots(12),$$

in which $\dfrac{d\lambda}{ds}=\dfrac{d}{ds}\left[\rho\left(\dfrac{dy}{ds}\cdot\dfrac{d^2z}{ds^2}-\dfrac{dz}{ds}\cdot\dfrac{d^2y}{ds^2}\right)\right]\ldots(a)$.

$$\frac{d\mu}{ds}=\frac{d}{ds}\left[\rho\left(\frac{dz}{ds}\cdot\frac{d^2x}{ds^2}-\frac{dx}{ds}\cdot\frac{d^2z}{ds^2}\right)\right]\ldots(b),$$

$$\frac{d\nu}{ds}=\frac{d}{ds}\left[\rho\left(\frac{dx}{ds}\cdot\frac{d^2y}{ds^2}-\frac{dy}{ds}\cdot\frac{d^2x}{ds^2}\right)\right]\ldots(c).$$

From (4), (5), and (6), by means of (7), we deduce

$$\frac{dx}{ds}=-\cos\omega\sin\theta\ldots(13),\ \frac{dy}{ds}=\cos\omega\cos\theta\ldots(14),\text{ and }\frac{dz}{ds}=\sin\omega\ \ .\ (15).$$

Reducing (a), (b), and (c), by means of (7), (8), (9), (10), (13), (14), (15); and then differentiating the results, we have respectively

$$\frac{d\lambda}{ds}=\frac{\sin\omega\cos\omega\cos\theta}{r}\ldots(16).\quad\frac{d\mu}{ds}=\frac{\sin\omega\cos\omega\cos\theta}{r}\ \ldots(17),$$

and $\dfrac{d\nu}{ds}=\dfrac{d}{ds}\left[\dfrac{r\cos^3\omega(\sin^2\theta+\cos^2\theta)}{\cos^2\omega}\right]=0\ldots(18)$.

Transforming (12) by means of (16), (17), and (18),

$$\frac{1}{\tau}=\sqrt{\left[\frac{\sin^2\omega\cos^2\omega\ (\sin^2\theta+\cos^2\theta)}{r^2}\right]}=\frac{\sin\omega\cos\omega}{r}\ldots(19).$$

which is also necessarily a *constant* quantity for every point of the curve.

NOTE.—Multiplying the numerator and the denominator of the right-hand member of (19), by $\cos\omega$, we have

$$\frac{1}{\tau}=\frac{\sin\omega}{\cos\omega}\times\frac{\cos\omega}{r}=\tan\omega\times\text{the curvature}.$$

If $\omega=\frac{1}{4}\pi$, the curvature and the tortuosity are necessarily equal. Had we assumed

$$x=\left(\frac{s}{\sqrt{6}}\right)\cos\left[\left(\frac{1}{\sqrt{2}}\right)\log\left(\frac{2s^2}{3c^2}\right)\right],\ y=\left(\frac{s}{\sqrt{6}}\right)\sin\left[\left(\frac{1}{\sqrt{2}}\right)\log\left(\frac{2s^2}{3c^2}\right)\right],$$

and $z=s\div\sqrt{2}$, then $1/\rho$ and $1/\tau$ would each have equaled $1/s$; that is, the curvature and the tortuosity would then have been the same for every point of the curve. Truly, the helix is a wonderful curve; it can easily be *axised* and *pitched* so as to have the same curvature and tortuosity as any given curve, while the loci of the centers of curvature and tortuosity are similar helices traceable on the same cylinder.

Also solved by *Professor G. B. M. Zerr.*

QUERIES AND INFORMATION.

RUSSIAN SCIENCE NOTES.

By Professor G. B. HALSTED, University of Texas, Austin, Texas.

The Jubilee-book issued by the University Kasan in commemoration of the Lobachevsky Centenary has already reached a very great circulation. His compatriots are pushing the non-Euclidean geometry.

N. P. Sokolov has just issued at Kiev (University Press) a pamplet of 32 pages (large 8vo.) entitled "The significance of the researches of N. I. Lobachevsky in geometry."

Volume IV of the second series of the Bulletin of the physico-mathematical society of Kasan, pp. 18-41, contains an interesting contribution by W. Sichstel on the fundamental theorems of spherical geometry.

Two books on America have lately been published in Russian. One is by Witkowsky, a scientist sent by the Russian government to study geodetic work in the United States. The other is published by a Russian, now resident in Los Angeles, who has been more than ten years in America, and has here amassed a fortune. He is a fervid republican, and writes under the *nom-de-plume* Tverski.

The well-known and justly admired writer Korolenko, ranked by the Russians second only to Tolstoi of living authors, was, during 1893, in America, and is about to issue his impressions of travel. This book, because of the high reputation of the author, is awaited with keen interest.

ARE LOBATSCHEWSKY'S PRINCIPLES APPLICABLE TO MECHANICS?

By WARREN HOLDEN, Professor of Mathematics, Pennsylvania Girard College, Philadelphia.

1. Is it expected that, when Lobatschewsky's new geometry is generally accepted, it will so permeate the old geometry as to modify its practical applications? Will the problems of engineering and construction have to be reconsidered?

Take a single example.

The builder of a railroad is very careful to keep the two rails of the track "everywhere equidistant." Unless they were parallel in the Euclidian sense disaster to a passing train would reveal the fact. Would Lobatschewsky's parallels, which always approach each other, answer for railroads?

2. Upon the surface of a sphere take a triangle with a given length of sides. The greater the diameter of the sphere the nearer the surface enclosed by the triangle approximates a plane, and simultaneously the nearer the sum of the angles of the triangle approaches two right angles. In passing from

surfaces, on which the angle sum is greater than two right angles, to surfaces, if any such there be, on which the angle sum is less than two right angles, we must come to a surface on which said sum is equal to two right angles. If this surface be not the Euclidian plane, what is it?

PROFESSOR SCHEFFER'S QUERY.

QUERY: Can any one furnish a reason for the peculiar name of the "devil's curve," or for the name which Professor Matz employs? J.F.W.S.

ANSWER.

"The Devil on Two Sticks" is best described as *a horizontal top*. According to the *Century Dictionary*, these toys are turned out of hard wood. Inclose the figure 8 with braces, $\{ 8 \}$; take a strong cord about four feet in length, and fasten one end of this cord in the concave part of each cusp of the braces. These braces are *distorted* branches of "la curbe du diable;" and they represent the *sticks*, by means of which the cord is rapidly manipulated in spinning the devil. Loop the cord around the waist of the devil; then bring the two sticks into a vertical plane; and acquire skill in balancing the devil and getting him into a motion of rotation about a vertical axis, by means of the friction of the cord. Unless you *spin* the devil in a *lively* manner, he will fall off the chord and hurt your toes in revenge! As to the genesis and history of this curve and its branches, all appears to be obscure; Cramer may have given the name "la courbe du diable," since this curve clearly characterizes the *outline* and *symmetry* of the toy—"The Devil on Two Sticks." So far as is known to Professor William Woolsey Johnson and the writer, Cramer is the first to use the French name of this curve. Undoubtedly *Routh's Rigid Dynamics* and other standard works on *Rotation*—not omitting Professor Ziwet's excellent *Treatise on Theoretical Mechanics*, would furnish very desirable theoretical knowledge in mathematically spinning "The Devil on Two Sticks."—F. P. MATZ.

ARTHUR CAYLEY.

By Professor G. B. HALSTED, University of Texas, Austin, Texas

How Professor Cayley touched every thing mathematical, and touched nothing which he did not adorn, may be illustrated by the following unpublished letters, which were the first expression of discoveries that have since taken their permanent place in our best text-books.

They are both the outcome of the sudden and fruitful interest in *linkage*, dating from Sylvester's interview with Tchebychev, when, leaving behind him the diagram of the now celebrated Peaucellier's Cell, the illustrious Russian gave in parting the characteristic advice: "Take to Kinematics, it will

repay you; it is more fecund than geometry; it adds a fourth dimension to space."

I will transcribe the letters exactly, not only because the recent death of Tchebychev, followed in less than two months by that of Cayley, gives them now a special pertinence, but because it is of interest to compare one with what is given on "tram motion" in Kempe's "How to Draw a Straight Line," and the other with its reproduction by no less a master than Clifford on pages 149, 150 of his Dynamic, whence I add figure 2.

"Roberts' theorem of 3 bar motion takes the following elegant form:

Take a triangle ABC & a point O and thro' O draw lines ∥ to the sides as in the figure—the 3 shaded △'s are of course similar to ABC.

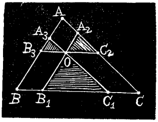


Fig. 1.

Now imagine a linkage composed of the shaded △'s and the bars AA_2, AA_3, BB_3, BB_1, CC_1, CC_2 pivotted together at A, B, C, A_2, A_3, B_3, B_1, C_2, C_1, O: then however the figure is moved, [of cource A_3, B_3 do not continue in the line AB, &c.], the triangle ABC *will remain similar to the shaded triangles*: and if in any position of the figure we fix the points A, B, C, then the point O will be moveable in a curve, viz. we have the *same* curve described by O considered as the vertex of OA_3B_3, where the two radii are AA_3, BB_3—by O considered as the vertex of OA_2C_2 &c.—and by O considered as the vertex of OB_1C_1, &c. Cambridge 22nd, Feby. 1876.

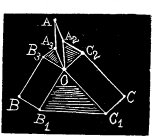


Fig. 2.

The porism is *very* pretty: it was new to me, tho' I think it ought not to have been so.

Look at the theorem thus. Imagine a plane, 2 points thereof A, C moving in fixed lines $O\alpha$, $O\gamma$.

Describe the circle OAC, which consider as a circle fixed in the plane & moveable with it. Then the theorem is that any point B of this circle moves in a line $O\beta$ thro' O.

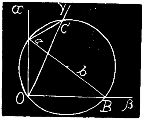


Fig. 3.

In particular B may be the opposite extremety of the diameter thro' A: and we have then the points A, B moving on the lines $O\alpha$, $O\beta$ at right angles to each other. Viz. the general case of a plane moving two points thereof on two fixed lines is reduced to this well-known particular case. And the theorem comes to this, that dividing the rod AB at pleasure into two parts AM, MB, and drawing MC at rt. angles and a mean proportional, the locus of C is a right line thro' O, which is of course easily proved."

Yours very sincerely, A. Cayley.

Cambridge, 5th May.

EDITORIALS.

THIS issue of the MONTHLY was mailed March 20th.

PERSONS failing to receive the MONTHLY shortly after the 25th of each month, should notify the PUBLISHERS at once.

OUR contributors will please send the material for the several departments to the editor of the respective departments and thus avoid delay and insure proper credit.

CARE should be taken in sending money in payment of subscriptions. The editors can not be responsible for money sent in any way other than by money order, draft, express money order, or registered letter. All money, drafts, etc., should be sent to B. F. FINKEL, Kidder, Missouri.

THROUGH an oversight, Dr. Matz was not credited with two different solutions of problem 11, Miscellaneous Department, nor with two different solutions of problem 26, Department of Calculus.

WE are pleased to note PROF. P. H. PHILBRICK is recovering from the effects of a severe attack of La Grippe.

DEPARTMENT F.—Mathematics of the University Extension Summer Meeting of the University of Pennysivania will be under the direction of Dr. I. J. Schwatt of the University of Pennsylvania. The meeting will be held from July 1st to July 27th. Classes will be formed in all departments of mathematics. On Monday evening July 1st, Dr. Schwatt will deliver to the students of all departments of the summer meeting, an address on The Importance of a Mathematical Training to Students of Various Branches of Science.

THROUGH the kindness of Dr. Halsted we have received five addresses delivered before the Texas Academy of Science. These addresses are given by men who are eminent in their special departments of science, and are both interesting and instructive.

WASHINGTON, D. C., Feb. 18, 1895.

PROF. B. F. FINKEL, A. M.,

MY DEAR SIR: Enclosed you will find a money order of \$3.00, which you will please accept as my subscription of the AMERICAN MATHEMATICAL MONTHLY for 1895. It is worth even more than that; and I fully agree with Prof. Matz. (a native of the same state and county with me), that it is almost impossible to give so much mathematics for \$2.00.

With best wishes for its success, and hoping that by the end of 1895 it shall have been put on a paying basis, I remain, with pleasure, its regular devoted reader and occasional contributor, M. A. GRUBER.

We have taken the privilege of publishing Mr. Gruber's letter in full, as a specimen of some of the encouraging words that come to the MONTHLY from time to time.